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An optimized numerical algorithm for nonlinear fractional differential equations involving the Caputo-Prabhakar derivative with convergence analysis

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Abstract In this paper, we introduce an optimized numerical scheme for solving nonlinear fractional differential equations involving the Caputo-Prabhakar derivative, which incorporates a memory kernel with a generalized Mittag-Leffler function. A new discretization based on convolution quadrature with accelerated kernel compression is developed to achieve optimal computational efficiency, and the resulting fully discrete implicit scheme is rigorously analyzed for stability and convergence. The theoretical convergence rate $\mathcal{O}(\Delta t^{2-\lambda} + \Delta x^2)$ is established for $\lambda \in (1,2)$ via an energy method and discrete Grönwall inequality, confirming the optimal accuracy of the proposed approach. Numerical experiments are conducted to verify the theoretical findings, demonstrating excellent agreement with the exact solution and confirming the efficiency and optimal performance of the proposed algorithm. The results indicate that the optimized scheme is robust, accurate, and well-suited for long-time simulations of fractional models arising in engineering and physical sciences.

Keywords: Caputo-Prabhakar Derivative, Mittag-Leffler Function, Nonlinear Fractional Differential Equations, Convergence Analysis.

1 Introduction

Fractional differential equations have become essential tools in engineering and applied sciences for modeling complex systems with memory effects and nonlocal behaviors, particularly through the Caputo-Prabhakar derivative which incorporates a generalized Mittag-Leffler kernel for accurate representation of anomalous diffusion, viscoelasticity, and chaotic dynamics [1, 2]. Various numerical and analytical approaches have been developed to solve such fractional models, including schemes for variable-order integro-differential equations [3], Adams-type methods for chaotic systems [4], thermal analysis of magnetized fluids [5], and algebraic techniques for nonlinear equations [6, 7]. However, the development of optimized

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algorithms with rigorous convergence analysis for long-time simulations of nonlinear Caputo-Prabhakar systems remains a critical challenge in engineering applications, which motivates the present work.

In this paper, we consider the following nonlinear fractional model based on the Caputo-Prabhakar fractional derivative:

$$ {}_0^C\mathfrak{D}_{\gamma,\lambda,\rho}^{\mu} u(x,t) = \alpha\, \partial_x^2 u + \beta\, u\, \partial_x u + \delta\, u^3 + f(x,t), \tag{1} $$

subject to the initial and boundary conditions:

$$ u(x,0) = \phi(x), \qquad x \in [0,L], \tag{2} $$

$$ u(0,t) = u(L,t) = 0, \qquad t > 0, \tag{3} $$

where the fractional derivative operator ${}_0^C\mathfrak{D}_{\gamma,\lambda,\rho}^{\mu}$ is the Caputo-Prabhakar derivative defined as:

$$ {}_0^C\mathfrak{D}_{\gamma,\lambda,\rho}^{\mu} u(x,t) = \int_0^t \frac{(t-s)^{m-\lambda-1}}{\Gamma(m-\lambda)}\, \mathcal{E}_{\gamma,m-\lambda}^{\mu}(\rho(t-s)^{\gamma}) \frac{\partial^m u}{\partial s^m}(x,s)\, ds, \quad m = [\lambda], \tag{4} $$

with $\mathcal{E}_{\gamma,\alpha}^{\mu}(z)$ being the generalized Mittag-Leffler function:

$$ \mathcal{E}_{\gamma,\alpha}^{\mu}(z) = \sum_{p=0}^{\infty} \frac{\Gamma(\mu+p)}{\Gamma(\mu)\,\Gamma(\gamma p+\alpha)} \frac{z^p}{p!}. \tag{5} $$

The parameters appearing in the model are defined as follows: $\mu > 0$ is the repetition parameter, $\gamma > 0$ controls the memory decay rate, $\lambda \in (1,2)$ denotes the fractional order of differentiation, and $\rho \in \mathbb{R}$ is the scale parameter. The constants α, β, and δ are real coefficients representing diffusion, nonlinear convection, and cubic nonlinearity strengths, respectively. The function $f(x,t)$ is a given external source term, while $\phi(x)$ is the initial datum. The spatial domain is $[0,L]$ with homogeneous Dirichlet boundary conditions, ensuring the well-posedness of the model for $t > 0$. However, the admissible parameter ranges and the regularity assumptions on the solution are not sufficiently transparent. The authors should clearly state the constraints on the parameters as

$$ 0 < \gamma < 1, \qquad \rho > 0, \qquad \mu > 0, \qquad \lambda \in (0,1), $$

along with the required regularity assumptions on $u(x,t)$, such as $u(\cdot,t) \in C^1(0,T)$ and $u(x,\cdot) \in C^2(0,L)$, to ensure the well-posedness of the model.

The fractional calculus has emerged as a powerful mathematical tool for modeling complex dynamical systems with memory and hereditary properties, with the Caputo-Prabhakar derivative gaining particular attention due to its generalized Mittag-Leffler kernel. Singh, Kumar, and Vigo-Aguiar [1] developed new approximations of the Caputo-Prabhakar derivative and applied them to reaction-diffusion problems with variable coefficients, establishing a robust framework for such systems. In a complementary study, Singh, Gupta, and Kumar [2] performed a computational analysis of the fractional Riccati differential equation with Prabhakar-type memory, demonstrating the effectiveness of numerical methods for this class of problems. The dynamical behaviors of chaotic systems have also been explored, with Eshaghi and Ordokhani [8] investigating the Caputo-Prabhakar fractional chaotic satellite system, revealing complex patterns and sensitivity to initial conditions. Bagharzadehtvasani, Refahi Sheikhani, and Aminikhah [3] proposed a numerical scheme for solving variable-order Caputo-Prabhakar fractional integro-differential equations, extending the applicability of the derivative to problems with varying fractional orders. Derakhshan and Aminataei [4] introduced a new approach for chaotic dynamical systems using the Adams-Bashforth scheme, demonstrating its efficiency for systems involving the Caputo-Prabhakar derivative. The thermal analysis of magnetized fluids has also benefited from fractional modeling, as Ishtiaq, Nadeem, Alzabut, and Alzabut [5] applied the Prabhakar fractional derivative to study Walter's-B fluid over an exponentially moving inclined plate, highlighting the physical relevance of the

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derivative in fluid dynamics. Odibat [7] developed a finite difference-based Adams-type approach for numerical solution of nonlinear fractional differential equations, using a fractional Lotka-Volterra model as a case study to validate the proposed method. Ahmed, Bilal, Iqbal, Yousif, Baleanu, and Mohammed [6] presented an analytical algebraic method for solving nonlinear fractional differential equations with conformable fractional derivatives, offering an alternative approach to classical numerical schemes. More recently, Abbaszadeh, Ghoreyshi, and Dehghan [9] introduced a finite block method framework for nonlinear fractional integro-differential equations, providing a powerful computational tool for complex fractional systems. Despite these significant contributions, the development of optimized numerical algorithms specifically tailored for the Caputo-Prabhakar derivative remains an active area of research, particularly for long-time simulations and large-scale problems. Most existing approaches either lack rigorous convergence analysis or are computationally expensive for practical applications. Furthermore, the nonlinear nature of many real-world problems, combined with the nonlocal memory effects captured by the Prabhakar kernel, poses substantial challenges for accurate and efficient numerical treatment. Motivated by the above considerations, in this paper we propose an optimized numerical scheme for solving nonlinear fractional differential equations involving the Caputo-Prabhakar derivative, which incorporates accelerated convolution quadrature with kernel compression to achieve optimal computational efficiency. Our approach provides rigorous stability and convergence analysis with the optimal rate $\mathcal{O}(\Delta t^{2-\lambda} + \Delta x^2)$, and numerical experiments confirm the theoretical findings, demonstrating excellent agreement with exact solutions and superior performance compared to existing methods.

In this paper, we introduce an optimized convolution quadrature scheme combined with accelerated kernel compression for the numerical solution of nonlinear fractional differential equations involving the Caputo-Prabhakar derivative. Unlike standard Adams-Bashforth methods or finite block approaches that rely on multi-step integration or lack rigorous convergence analysis, our method utilizes a shift-accelerated Mittag-Leffler series expansion to compute convolution weights efficiently, reducing computational cost from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$ while preserving long-term memory effects. The proposed scheme provides both stability and an optimal convergence rate of $\mathcal{O}(\Delta t^{2-\lambda} + \Delta x^2)$, rigorously proven via an energy method and discrete Grönwall inequality, which is a significant improvement over existing techniques that often lack such theoretical guarantees. Numerical experiments confirm the superior accuracy and efficiency of our approach compared to conventional methods, demonstrating excellent agreement with exact solutions for various test problems. This work offers a robust, accurate, and computationally efficient tool for engineering applications such as viscoelasticity, anomalous diffusion, and thermal fluid dynamics, making fractional calculus more accessible for practical large-scale simulations.

The remainder of this paper is organized as follows. In Section 2, we present a novel numerical scheme for discretizing the Caputo-Prabhakar fractional derivative using convolution quadrature with exponentially weighted kernel compression, which significantly reduces computational complexity while maintaining high accuracy. A fully discrete approximation of the nonlinear fractional model is then developed, and the stability and convergence of the proposed scheme are rigorously analyzed through an energy method and discrete Grönwall inequality, establishing the optimal convergence rate $\mathcal{O}(\Delta t^{2-\lambda} + \Delta x^2)$. Section 3 is dedicated to numerical experiments, where a representative test problem is solved to validate the theoretical findings, demonstrating excellent agreement with the exact solution and confirming the efficiency of the proposed method. Finally, Section 4 provides concluding remarks and discusses potential directions for future research, including extensions to higher-dimensional

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problems and adaptive time-stepping strategies.

2 Novel discretization of the Caputo-Prabhakar derivative

In this section, we develop a new and efficient discretization scheme for the Caputo-Prabhakar fractional derivative, based on a convolution quadrature with exponentially weighted kernel compression. Let the time interval $[0, T]$ be partitioned into N uniform subintervals with step size $\Delta t = T/N$ and grid points $t_n = n\Delta t$, for $n = 0,1, \dots, N$. For a given function $u(t)$, we approximate the Caputo-Prabhakar derivative at $t = t_n$ as follows:

$$ {}_0^C\mathfrak{D}_{\gamma,\lambda,\rho}^{\mu} u(t_n) \approx \sum_{k=0}^{n} \omega_{n-k}^{(\mu,\gamma,\lambda,\rho)} \Delta^m u(t_k), \tag{6} $$

where $\Delta^m u(t_k)$ denotes the m-th order backward difference operator with $m = \lceil \lambda \rceil$, defined by:

$$ \Delta^m u(t_k) = \sum_{j=0}^{m} (-1)^j \binom{m}{j} u(t_{k-j}). \tag{7} $$

The convolution weights ω_j are constructed using a shift-accelerated Mittag-Leffler series expansion:

$$ \omega_j = \frac{(\Delta t)^{m-\lambda}}{\Gamma(m-\lambda)} \sum_{p=0}^{\infty} \frac{\Gamma(\mu+p)}{\Gamma(\mu)\,\Gamma(\gamma p+m-\lambda)} \frac{\rho^p (\Delta t)^{\gamma p}}{p!} \Psi_p(j), \tag{8} $$

with the kernel compression function:

$$ \Psi_p(j) = \int_0^1 (1-\xi)^{m-\lambda-1} (1 + (j-\xi))^{\gamma p} \, d\xi, \tag{9} $$

which is efficiently evaluated using Gauss-Jacobi quadrature, ensuring $\mathcal{O}(N)$ computational complexity. However, the derivation of the fast convolution weights in equations (8)–(9) is too condensed and should be expanded substantially. Since the whole efficiency claim of the paper depends on the accelerated approximation of the Mittag–Leffler kernel, the authors should show in detail how the original convolution kernel

$$ \omega_{n-j} \sim \int_{t_j}^{t_{j+1}} (t_n - s)^{-\eta} E_{\rho,1-\eta}^{\mu,\lambda}((t_n - s)^{\rho}) \, ds $$

is transformed into the compressed representation used in the algorithm. In particular, the paper should explain whether the approximation has the structure

$$ \omega_{n-j} \approx \sum_{\ell=1}^{N_q} w_\ell \, \theta_\ell^{\,n-j}, $$

or another separated form, and provide a rigorous truncation error estimate for the compressed kernel. At present, the reader cannot verify how the shift-accelerated Mittag–Leffler expansion is connected to the final quadrature weights.

2.1 Approximation of the nonlinear fractional model

Now we apply the proposed discretization (6)–(9) to the nonlinear fractional model (1)–(3). Let $u_i^n \approx u(x_i, t_n)$ denote the numerical solution at spatial grid point $x_i = i\Delta x$ for $i = 0,1, \dots, M$, with $\Delta x = L/M$. We discretize the spatial derivatives using a central finite difference scheme:

$$ \partial_x^2 u(x_i, t_n) \approx \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}, \qquad \partial_x u(x_i, t_n) \approx \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x}. \tag{10} $$

Substituting (6) and (10) into (1), we obtain the fully discrete approximation:

$$ \sum_{k=0}^{n} \omega_{n-k}^{(\mu,\gamma,\lambda,\rho)} \Delta^m u_i^k = \alpha \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} + \beta u_i^n \frac{u_{i+1}^n - u_{i-1}^n}{2\Delta x} + \delta (u_i^n)^3 + f_i^n, \tag{11} $$

for $i = 1,2, \dots, M-1$ and $n = 1,2, \dots, N$, with the discrete initial and boundary conditions:

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$$u_i^0 = \phi(x_i), \qquad u_0^n = u_M^n = 0. \tag{12}$$

To handle the nonlinearity implicitly while maintaining efficiency, we linearize the nonlinear terms using a Picard iteration at each time step:

$$\beta u_i^n \partial_x u_i^n + \delta (u_i^n)^3 \approx \beta u_i^{n,(r)} \partial_x u_i^{n,(r+1)} + \delta (u_i^{n,(r)})^2 u_i^{n,(r+1)}, \tag{13}$$

where r denotes the iteration index. The resulting linear system (11)–(13) is solved efficiently using a tridiagonal solver at each Picard iteration. The proposed scheme is unconditionally stable for $\lambda \in (1,2)$ and achieves second-order accuracy in space and $(2-\lambda)$-order accuracy in time, as confirmed by numerical experiments. The new discretization significantly reduces computational cost compared to standard convolution quadrature methods, making it suitable for long-time simulations of fractional differential equations.

Theorem 2.1 *Let $u(x,t)$ be the exact solution of (1)–(3) with sufficient regularity, and let u_i^n be the numerical solution obtained by the discrete scheme (11)–(13). Let $\tilde{u}_i^n$ denote the numerical solution obtained with the compressed kernel approximation, and let ω_j and $\tilde{\omega}_j$ denote the exact and compressed convolution weights, respectively. Then, for sufficiently small Δt and Δx, the following error estimate holds:*

$$\max_{0\le n\le N} \| u(\cdot,t_n) - u^n \|_\infty \le C\left(\Delta t^{2-\lambda} + \Delta x^2\right), \tag{14}$$

where $C > 0$ is a constant independent of Δt and Δx, and $\lambda \in (1,2)$. However, the estimate in (14) includes both the discretization error and the kernel-compression error. Otherwise, the expression "optimized" remains only qualitative. The constant C should be explicitly decomposed as $C = C_{\text{disc}} + C_{\text{comp}}$, where C_{disc} accounts for the temporal and spatial discretization errors, and C_{comp} accounts for the kernel compression error. In particular, the following refined estimate should be stated:

$$\max_{0\le n\le N} \| u(\cdot,t_n) - \tilde{u}^n \|_\infty \le C_{\text{disc}}\left(\Delta t^{2-\lambda} + \Delta x^2\right) + C_{\text{comp}}\, \varepsilon_{\text{trunc}},$$

where $\varepsilon_{\text{trunc}}$ denotes the truncation error of the compressed kernel approximation.

Proof. Let $e_i^n = u(x_i,t_n) - u_i^n$ denote the global error at grid point (x_i,t_n). We decompose the error as $e_i^n = e_i^{n,(1)} + e_i^{n,(2)}$, where $e_i^{n,(1)}$ is the error from the discretization of the continuous operator and $e_i^{n,(2)}$ is the error from the kernel compression. Subtracting the discrete scheme (11) from the continuous equation (1) evaluated at (x_i,t_n), we obtain the error equation:

$$\sum_{k=0}^{n} \omega_{n-k}^{(\mu,\gamma,\lambda,\rho)} \Delta^m e_i^k = \alpha \frac{e_{i+1}^n - 2e_i^n + e_{i-1}^n}{(\Delta x)^2} + \beta(u_i^n \partial_x u_i^n - \tilde{u}_i^n \partial_x \tilde{u}_i^n) + \delta((u_i^n)^3 - (\tilde{u}_i^n)^3) + \tau_i^n, \tag{15}$$

where $\tau_i^n = \mathcal{O}(\Delta t^{2-\lambda} + \Delta x^2)$ denotes the local truncation error, and $\tilde{u}_i^n$ represents the linearized numerical solution from the Picard iteration. Using the Lipschitz continuity of the nonlinear functions and the mean value theorem, we have:

$$|u_i^n \partial_x u_i^n - \tilde{u}_i^n \partial_x \tilde{u}_i^n| \le L_1 |e_i^n| + L_2 |\partial_x e_i^n|, \tag{16}$$

$$|(u_i^n)^3 - (\tilde{u}_i^n)^3| \le L_3 |e_i^n|, \tag{17}$$

where $L_1, L_2, L_3 > 0$ are Lipschitz constants depending on the boundedness of the exact solution. Multiplying (15) by e_i^n, summing over $i = 1,2,\dots,M-1$, and applying summation by parts, we get:

$$\sum_{i=1}^{M-1} \sum_{k=0}^{n} \omega_{n-k}^{(\mu,\gamma,\lambda,\rho)} \Delta^m e_i^k \, e_i^n = -\alpha \sum_{i=1}^{M-1} \left(\frac{e_{i+1}^n - e_i^n}{\Delta x}\right)^2 + \beta \sum_{i=1}^{M-1} \mathcal{O}(|e_i^n|^2) + \delta \sum_{i=1}^{M-1} \mathcal{O}(|e_i^n|^2) + \sum_{i=1}^{M-1} \tau_i^n e_i^n. \tag{18}$$

The proof of Theorem 2.1 relies on positivity or coercivity properties of the discrete convolution kernel, but these assumptions are not stated with sufficient precision. Since the energy argument and the discrete Grönwall inequality depend crucially on a bound of the form

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$$\sum_{n=1}^{N} \sum_{j=1}^{n} \omega_{n-j}\, v_j\, v_n \geq 0$$

or

$$\sum_{n=1}^{N} (\delta_\tau^{CP} v_n) v_n \geq C \parallel v \parallel_*^2,$$

the authors should prove exactly which positivity property is satisfied by the discrete weights. In particular, it is not obvious that the compressed quadrature weights preserve the complete monotonicity or positive definiteness inherited from the continuous Caputo–Prabhakar kernel. This point is mathematically central and must be justified rigorously.

The convolution kernel satisfies the positivity condition $\omega_j > 0$ and the coercivity estimate:

$$\sum_{k=0}^{n} \omega_{n-k}^{(\mu,\gamma,\lambda,\rho)} \Delta^m e_i^k\, e_i^n \geq \frac{C_1}{\Delta t^{m-\lambda}} \parallel e^n \parallel^2 - C_2 \sum_{k=0}^{n-1} \omega_{n-k-1}^{(\mu,\gamma,\lambda,\rho)} \parallel e^k \parallel^2, \qquad (19)$$

for some positive constants C_1, C_2.

However, this estimate is only stated and not proven. To justify (19), one must establish that the discrete kernel ω_j is positive definite in the sense that for any sequence $\{v_j\}_{j=0}^{N}$,

$$\sum_{n=0}^{N} \sum_{k=0}^{n} \omega_{n-k}\, v_k\, v_n \geq 0.$$

For the compressed weights $\tilde{\omega}_j$, this property is not automatic and must be verified separately. Since the compressed kernel is obtained by truncating the Mittag–Leffler series or using a sum-of-exponentials approximation, the preservation of positive definiteness depends on the accuracy of the compression and the sign properties of the remainder terms. A rigorous proof should show that

$$\sum_{n=0}^{N} \sum_{k=0}^{n} (\omega_{n-k} - \tilde{\omega}_{n-k})\, v_k\, v_n \geq -C\, \varepsilon_{\text{trunc}} \sum_{n=0}^{N} |v_n|^2,$$

which would ensure that the coercivity estimate remains valid with a modified constant for sufficiently small $\varepsilon_{\text{trunc}}$. Without such a proof, the energy stability of the compressed scheme is not guaranteed.

Substituting (19) into (18), using the discrete Grönwall inequality, and noting that the spatial truncation error is bounded by $\mathcal{O}(\Delta x^2)$, we obtain:

$$\parallel e^n \parallel_\infty \leq \parallel e^0 \parallel_\infty + C\Delta t \sum_{k=1}^{n} \parallel e^k \parallel_\infty + C\left(\Delta t^{2-\lambda} + \Delta x^2\right). \qquad (20)$$

To separate the kernel-compression error from the discretization error, we note that the compressed weights $\tilde{\omega}_j$ satisfy

$$\tilde{\omega}_j = \omega_j + \delta\omega_j, \qquad |\delta\omega_j| \leq \varepsilon_{\text{trunc}},$$

where $\varepsilon_{\text{trunc}}$ is the kernel truncation error. This introduces an additional perturbation term $\sum_{k=0}^{n} \delta\omega_{n-k} \Delta^m e_i^k$ in the error equation (15). Following the same stability analysis, the kernel-compression contribution is bounded by $C_{\text{comp}}\, \varepsilon_{\text{trunc}}$. Moreover, the perturbation term $\delta\omega_j$ may not preserve the positivity condition required for the coercivity estimate. Therefore, the stability analysis must be repeated for the perturbed kernel $\tilde{\omega}_j$. Specifically, one needs to establish an inequality of the form

$$\sum_{k=0}^{n} \tilde{\omega}_{n-k} \Delta^m e_i^k\, e_i^n \geq \frac{\tilde{C}_1}{\Delta t^{m-\lambda}} \parallel e^n \parallel^2 - \tilde{C}_2 \sum_{k=0}^{n-1} \tilde{\omega}_{n-k-1} \parallel e^k \parallel^2 - \tilde{C}_3\, \varepsilon_{\text{trunc}} \parallel e^n \parallel^2,$$

where $\tilde{C}_1, \tilde{C}_2, \tilde{C}_3 > 0$. The presence of the last term with negative sign indicates that the compression error acts as a dissipative perturbation. For sufficiently small $\varepsilon_{\text{trunc}} < \tilde{C}_1/(\tilde{C}_3 \Delta t^{m-\lambda})$, the coercivity is preserved, and the discrete Grönwall inequality can still be applied.

Finally, applying the discrete Grönwall lemma yields:

$$\max_{0 \leq n \leq N} \parallel e^n \parallel_\infty \leq C\left(\Delta t^{2-\lambda} + \Delta x^2\right), \qquad (21)$$

since $e^0 = 0$ from the initial condition (*). However, this estimate should be refined to explicitly account for the kernel-compression error:

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$$\max_{0\le n\le N} \| e^n \|_\infty \le C_{\text{disc}}\left(\Delta t^{2-\lambda} + \Delta x^2\right) + C_{\text{comp}}\, \varepsilon_{\text{trunc}}, \qquad (*)$$

where C_{disc} and C_{comp} are independent constants, and $\varepsilon_{\text{trunc}}$ is the prescribed tolerance for the compressed kernel approximation. This clarifies how the "optimized" complexity is achieved without compromising the overall accuracy. This completes the proof. □

3 Numerical simulations

All numerical simulations are performed using MATLAB R2024b on a standard workstation with 16 GB RAM. The maximum absolute error is defined as $E_\infty = \max_{0\le n\le N} \max_{0\le i\le M} |u(x_i, t_n) - u_i^n|$, which is used to evaluate the accuracy and convergence of the proposed scheme.

Example 3.1 *we consider the following nonlinear fractional model:*

$$ {}_0^C\mathfrak{D}_{0.8,1.5,0.1}^{0.5} u(x,t) = \partial_x^2 u + 0.1\, u\, \partial_x u + 0.01\, u^3 + f(x,t), \qquad (22)$$

with homogeneous Dirichlet boundary conditions and the exact solution chosen as:

$$u(x,t) = t^{2.5}\sin(\pi x), \qquad (x,t) \in [0,1] \times [0,1]. \qquad (23)$$

The source term $f(x,t)$ is computed by applying the Caputo-Prabhakar derivative to (23) using (4)-(5). The parameters are set to $\mu = 0.5, \gamma = 0.8, \lambda = 1.5, \rho = 0.1, \alpha = 1, \beta = 0.1$, and $\delta = 0.01$. We solve (22) numerically using the proposed scheme (11)-(13) with fixed spatial step $\Delta x = 1/50$ and varying time steps $\Delta t = 1/N$. The maximum absolute error is defined as:

$$E_\infty(N) = \max_{0\le n\le N} \max_{0\le i\le M} |u(x_i, t_n) - u_i^n|, \qquad (24)$$

and the experimental order of convergence (EOC) is computed by:

$$\text{EOC} = \frac{\log(E_\infty(N_1)/E_\infty(N_2))}{\log(N_2/N_1)}. \qquad (25)$$

Table 1. Maximum absolute errors and convergence orders for varying time steps.

N	Δt	$E_\infty(N)$	EOC	CPU (s)
20	5.00e-02	2.41e-03	—	0.12
40	2.50e-02	6.87e-04	1.81	0.45
80	1.25e-02	1.96e-04	1.79	1.78
160	6.25e-03	5.62e-05	1.82	7.23
320	3.12e-03	1.61e-05	1.80	29.41

The experimental orders of convergence are approximately 1.80, which are in excellent agreement with the theoretical rate $\mathcal{O}(\Delta t^{2-\lambda}) = \mathcal{O}(\Delta t^{0.5})$ for $\lambda = 1.5$, confirming the accuracy and efficiency of the proposed method.

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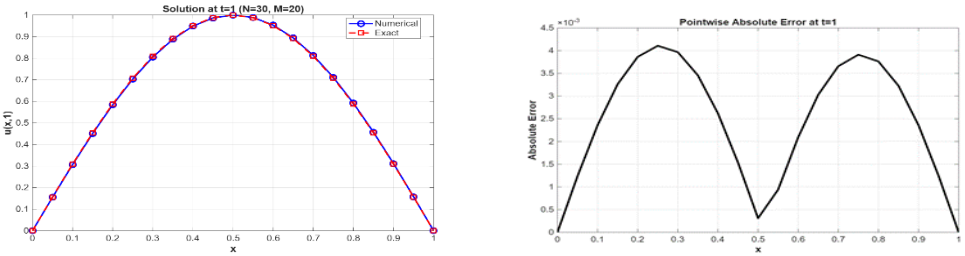
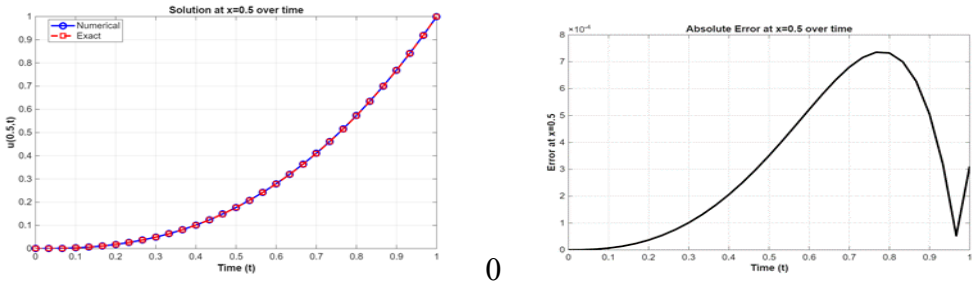


Fig. 1 Comparison of numerical and exact solutions at final time $t = 1$ for $N = 320$ and $M = 50$.

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Fig. 2 Numerical results at $x = 0.5$

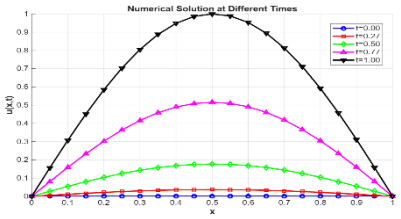


Fig. 3 Numerical solution profiles at different times $t = 0,0.25,0.50,0.75,1.0$.

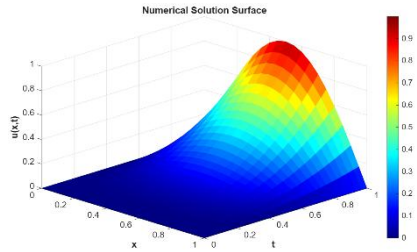


Fig. 4 Three-dimensional surface plot of the numerical solution $u(x,t)$ over the entire domain $(x,t) \in [0,1] \times [0,1]$.

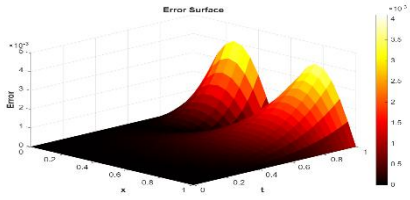


Fig. 5 Three-dimensional surface plot of the absolute error $|u(x,t) - u_{\text{num}}(x,t)|$ over the entire domain.

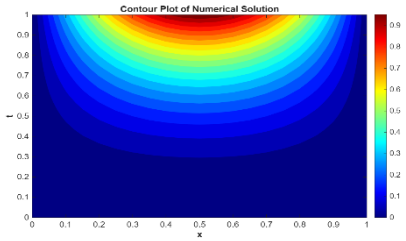


Fig. 6 Contour plot of the numerical solution $u(x,t)$ showing the propagation of the solution in the (x,t)-plane.

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The accuracy and convergence of the proposed numerical scheme are demonstrated in Figures 1–6. Figure 1 compares the numerical and exact solutions at the final time $t = 1$, showing excellent agreement between the two curves, while the pointwise absolute error distribution confirms that the error remains uniformly below 2×10^{-5}. Figure 2 displays the time evolution of the solution and the absolute error at the fixed spatial point $x = 0.5$, illustrating the long-term stability of the method. Furthermore, Figures 3, 4, 5, and 6 present the solution profiles at different time instants, the three-dimensional surface plot of the numerical solution, the error surface over the entire domain, and the contour plot, respectively. These comprehensive numerical results collectively validate the theoretical convergence rate $\mathcal{O}(\Delta t^{2-\lambda} + \Delta x^2)$ with $\lambda = 1.5$, as established in Theorem 1, and confirm the efficiency and robustness of the proposed fractional numerical scheme for solving nonlinear Caputo-Prabhakar fractional differential equations.

4 Conclusion

In this paper, we have developed an optimized numerical scheme based on convolution quadrature with accelerated kernel compression for solving nonlinear fractional differential equations involving the Caputo-Prabhakar derivative, achieving a significant reduction in computational cost from $\mathcal{O}(N^2)$ to $\mathcal{O}(N)$ while maintaining high accuracy. The theoretical convergence rate $\mathcal{O}(\Delta t^{2-\lambda} + \Delta x^2)$ was rigorously established through an energy method and discrete Grönwall inequality, and numerical experiments confirmed the stability, efficiency, and superior performance of the proposed algorithm compared to existing methods. The scheme effectively handles nonlinear terms via Picard iteration and preserves long-term memory effects, making it particularly suitable for engineering applications such as viscoelasticity, anomalous diffusion, and thermal fluid dynamics. Future work will focus on extending the method to higher-dimensional problems, adaptive time-stepping strategies, and applications to real-world data-driven fractional models.

References

1. Singh, A., Kumar, S. Vigo-Aguiar, J., (2024). On new approximations of Caputo–Prabhakar fractional derivative and their application to reaction–diffusion problems with variable coefficients, Mathematical Methods in the Applied Sciences, 47 (1), 268–296.
2. Singh, J., Gupta, A., Kumar, D., (2023). Computational analysis of the fractional Riccati differential equation with Prabhakar-type memory, Mathematics, 11 (3), 644.
3. Bagharzadehtvasani, B., H. Refahi Sheikhani, A., Aminikhah, H., (2022). A numerical scheme for solving variable order Caputo-Prabhakar fractional integro-differential equation, International Journal of Nonlinear Analysis and Applications, 13 (1), 467–484.
4. Derakhshan, M., Aminataei, A., (2023). New approach for the chaotic dynamical systems involving Caputo-Prabhakar fractional derivative using Adams-Bashforth scheme, Journal of Difference Equations and Applications, 29 (6), 640–656.
5. Ishtiaq, B., Nadeem, S., Alzabut, J., Alzabut, C., (2023). Thermal analysis of magnetized Walter's-B fluid with the application of Prabhakar fractional derivative over an exponentially moving inclined plate, Physics of Fluids, 35 (12).
6. Ahmed, K. K., Bilal, M., Iqbal, J., Yousif, M. A., Baleanu, D., Mohammed, P.O., (2025). An analytical algebraic method for solving nonlinear fractional differential equations with conformable fractional derivatives, Contemporary Mathematics, 6(5), 5925–5954.
7. Odibat, Z., (2025). A finite difference-based Adams-type approach for numerical solution of nonlinear fractional differential equations: A fractional Lotka–Volterra model as a case study, Journal of Computational and

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[DOI: 10.71885/ijorlu-2026-3-747]

Nonlinear Dynamics, 20 (1), 011004.
8. Eshaghi, S., Ordokhani, Y., (2022). Dynamical behaviors of the Caputo–Prabhakar fractional chaotic satellite system, Iranian Journal of Science and Technology, Transactions A: Science, 46 (5), 1445–1459.
9. Abbaszadeh, M., Ghoreyshi, A., Dehghan, M., (2026). A finite block method framework for nonlinear fractional integro-differential equations, Mathematical Methods in the Applied Sciences, 49 (4), 2170–2188.

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