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Measuring Interval Efficiency in Convex Hull of Data Using DEA

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Abstract In this paper a model is introduced to evaluate decision making units (DMUs) in the production possibility set (PPS), with three properties namely(a) The observation (x_j, y_j), j=1,...,n ,belongs to PPS. (b) Any convex combination of observation in PPS belongs to PPS. (c) PPS is the smallest set satisfying the principles (a) and (b). Then we studied the effect of imprecise data in the convex hull of DMUs model (CHD). We analyzed the efficiency frontier in CHD model with imprecise data. We have applied the first and second method in CHD model for interval data. We have presented first the effect of interval data in the CHD model. Finally, an example has been presented for analyzing the CHD model with interval data. In this example efficiency is calculated on the convex hull with first method and second method.

Keywords: Data Envelopment Analysis, Convex Hull, Efficient Frontier, Interval Data, Interval Efficiency.

1 Introduction

Data envelopment analysis (DEA) was initiated in 1978 by Charnes, Cooper and Rhodes (CCR) [1]. In DEA, the organizations under study are called decision making units (DMUs).Generically, a DMU is regarded as the entity responsible for converting inputs into outputs whose performance is going to be evaluated. DEA evaluates the efficiency of DMUs relative to the production possibilities, and moreover identifies reference units that can help to find out causes and remedies for inefficiencies. The efficiency of a DMU is a scalar measure ranging between zero and one. This scalar value is measured through a linear programming model ([1,2]).

Free Disposal Hull (FDH) models relies on the sole assumption that production possibilities satisfy free disposability and ensure that efficiency evaluations are effected from only actually observed performances.

Classic DEA models are not efficiency evaluation based on the slacks. One of the main

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objectives of DEA is to measure the efficiency of a DMU by a scalar measure ranging between zero (the worst) and one (the best). This scalar value is measured through a linear programming model. Charnes et al. [3] developed the additive model of DEA, which deals directly with input excesses and output shortfalls. This model has no scalar measure (ratio efficiency), although this model can discriminate between efficient and inefficient DMUs by the existence of slacks, it has no means of gauging the depth of inefficiency. In an attempt to define inefficiency based on the slacks, Tone [4], Russell[5], [6], Pastor [7], Lovell and Pastor [8], Torgersen et al. [9], Copper and Pastor [10], Copper and Tone [11], Thrall [12] and others have proposed several formulas for finding a scalar measure.

In classic DEA models, such as BCC and CCR models, two models have been presented that divided DMUs into two categories of strong efficient and non-strong efficient, in two phases. A long effort has been made to do that in one model. Finally, in 1990 Ali and Seiford[13],presented a model known as the additive model. They proved constancy of this model toward transformation. One of the problems of this model is dividing DMUs just in two categories of strong efficient and non-strong efficient. Also, it does not present any criterion for DMUs efficiency amount. To this end, scientists attempted to present a model which has not only the advantages of additive model in categorizing DMUs in two categories of strong efficient and non-strong efficient, but also it presents a criterion to efficiency amount. To this purpose, Tone [4] presented the SBM model which has both the above advantages. Then, Russell [5] presented a model which is equivalent to SMB model. Each of SMB or Russell models can be used in CHD model to assess efficiency of DMUs, but a model will be presented in the following according to ideal and anti-ideal DMUs to evaluate efficiency of DMUsin CHD model.

DEA, as a very useful management and decision tool, has found surprising development in theory and methodology and extensive applications in the range of the whole world since it was first developed by Charnes et al [1]. Traditional DEA models such as CCR and BBC models and so on do not deal with imprecise data and assume that all input and output data are exactly known. In real world situations, however, this assumption may not always be true. Due to the existence of uncertainty, DEA sometimes faces the situation of imprecise data, especially when a set of DMUs contains missing data, judgment data, forecasting data or ordinal preference information. Generally speaking, uncertain information or imprecise data can be expressed in interval or fuzzy numbers. Therefore, how to evaluate the management or operation efficiency of a set of DMUs in interval environments is a worth-studying problem. This is the need of both the developments of DEA theory and methodology and its real applications. Therefore, we have studied the effect of imprecise data in the CHD model. We have presented first the effect of interval data in the CHD model.

This paper is organized as follows. In Section 2, some discussions regarding to convex hull of DMUs are stated and our considered PPS and presents a model to recognize strong efficient DMUs and this model has a criterion to efficiency amount in the considered PPS. This model is based on the slacks and following according to ideal and anti-ideal DMUs. In Section 3 we will study effect of imprecise data in the CHD model. In Section 4we will present first method for efficiency interval in CHD model. In Section 5we will present the second efficiency interval in CHD model. To explain the accuracy of what have presented, an example is illustrated in Section 6. Finally, Section 7concludes the paper.

2 Convex hull of DMUs

Suppose we have pairs of positive input and output vectors (x_j, y_j) *for* each j=1,...,n of n DMUs. All data (x_j, y_j) *for* each j=1,...,n are assumed to be nonnegative but at least one component of every input and output vector is positive. We refer to this as semi positive with a mathematical characterization given by $x_j \geq \circ, x_j \neq \circ$ and $y_j \geq \circ, y_j \neq \circ$ that j=1,...,n . Therefore, each DMU is supposed to have at least one positive value in both input and output. We will call a pair such semi positive input $x \in R^m$ and output $y \in R^s$, and express them by the notation (x, y).The set of feasible observations is called the production possibility set (PPS).The production possibility set of CCR model is always called T_{CCR} .Charnes, Cooper and Rhodes in presenting CCR model to asses DMUs, accepted the following hypothesis for PPS.

1. The observation (x_j, y_j) *for* each j=1,...,n belongs to T_{CCR} .
2. If an (x, y) belongs to T_{CCR} , then (tx, ty) belongs to T_{CCR} for any positive scalar t. we call this property the constant returns-to-scale assumption.
3. For an (x, y) in T_{CCR} , any semi positive $(\overline{x}, \overline{y})$ with $(\overline{x} \geq x)$ and $(\overline{y} \leq y)$ is included in T_{CCR} . That is, any observation with input no less than x in any component and with output no greater than y in any component is feasible.
4. Any semi positive linear combination of observation in T_{CCR} belongs to T_{CCR} .
5. T_{CCR} is the smallest set that satisfying the principles 1 to 4.

Denoting the data sets in matrices $X = (x_j)$ and $Y = (y_j)$, we can define the production possibility set T_{CCR} satisfying 1 to 5 by

$$T_{CCR} = \{(x, y) | x \geq X\lambda, \ y \leq Y\lambda, \ \lambda \geq \circ\} \quad (1)$$

Where X is a semi positive vector in R^n [1].

Banker et al. [2] defined PPS with variable return to scale (VRS) with acceptance axioms 1, 3, 4 and 5. The model with variable return to scale is called BCC model. It considered PPS is as follows.

$$T_{BCC} = \{(x, y) | x \geq X\lambda, \ y \leq Y\lambda, 1\lambda = 1, \ \lambda \geq \circ\} \quad (2)$$

Soltanifar et al. [14] defined convex hull as follows:

Definition1. Let E be a nonempty subset in R^n. Then $CH(E)$ CH (E) is called the convex hull of E which isE components convex combination set and as follows:

$$CH(E) = \left\{X \in R^n : X = \sum_{j=1}^{n} \lambda_j x_j, \ \sum_{j=1}^{n} \lambda_j = 1, \ x_j \in E, \ \lambda_j \geq \circ\right\} \quad (3)$$

By definition 1, convex hull has definition ability on any nonempty subset in R^n. Convex hull is obtaining of itself set components convex combination.

Suppose an organization has n DMUs, produces s outputs denoted by y_{rj}, the r[th] output of DMU, for r=1,...,s and consumes m inputs denoted by x_{ij}, the i[th] input of DMU_j for i=1,...,m and j=1,...,n . we postulate the following properties of PPS:

1. The observation (x_j, y_j) *for* each j=1,...,n belong to T_{CHD}.
2. Any convex combination of observation in T_{CHD} belong to T_{CHD}.
3. T_{CHD} is the smallest set that satisfying the principles 1 and 2.

three properties above is called axiom principle of CHD model.
Then we can define the production possibility set T_{CHD} satisfying 1 to 3 by

$$T_{CHD} = \left\{(x, y) \middle| x = X\lambda,\ y = Y\lambda, 1\lambda = 1,\ \lambda \geq 0\right\} \quad (4)$$

where "CHD" means convex hull of DMUs. The following preliminary discussions are used in this paper:

Definition2. DMU_p denoted by (x_j, y_j) is named (strongly) convex efficient (c-efficient) if and only if there does not exist another point in convex hull of DMUs such as (x, y) subject to $(-x, y) \underset{\neq}{\geq} (-x_p, y_p)$.

Definition3. An IDMU is a virtual DMU, which can use the least inputs to generate the most outputs. While an ADMU is a DMU, which consumes the most inputs only to produce the least outputs.

According to above definition, we denote by $x_i^{\min}$ for r=1,...,s and $y_r^{\max}$ for r=1,...,s the inputs and outputs of the IDMU and by $x_i^{\max}$ for r=1,...,s and $y_r^{\min}$ for r=1,...,s The input and outputs of the ADMU respectively, where $x_i^{\min}$ and $x_i^{\max}$ are minimum and maximum of the i[th] input $y_r^{\min}$ and $y_r^{\max}$ are the minimum and maximum of the r[th] output[14].

Soltanifar et al. [14] defined CHD model as follows:

$$\rho_p^* = \min 1 - \frac{\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-}{\alpha + \varepsilon}$$

$$s.t. \tag{5}$$

$$\sum_{j=1}^{n} \lambda_j y_{rj} - s_r^+ = y_{rp}, \ \text{r=1,...,s}$$

$$\sum_{j=1}^{n} \lambda_j x_{ij} + s_i^- = x_{ip}, \ \text{i=1,...,m}$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq \circ, \ \text{j=1,...,n}$$

$$s_r^+ \geq \circ, \ s_i^- \geq \circ$$

$$where$$

$$\alpha = \left\| (x^{\min}, y^{\max}) - (x^{\max}, y^{\min}) \right\|_1 = \sum_{i=1}^{m} \left| x_i^{\min} - x_i^{\max} \right| + \sum_{r=1}^{s} \left| y_r^{\max} - y_r^{\min} \right| \geq \circ \tag{6}$$

ε is a positive and small enough number and $\|.\|_1$ representative of L_1 norm. Note that if production possibly set is not a singleton set then α will be positive. α is $\|.\|_1$ IDMU and ADMU. Therefore, CHD model is presented in the following according to ideal and anti-ideal DMUs.

Convex hull of DMUs contains efficiency frontier and inefficiency frontier. CHD model evaluates DMUs performance on the convex hull. In this model, that all DMUs located on the convex hull aren't efficient, rather the only DMUs located on efficiency frontier are efficient. DMUs located on inefficiency frontier are inefficient. Fig. 1 shows sample of CHD model.

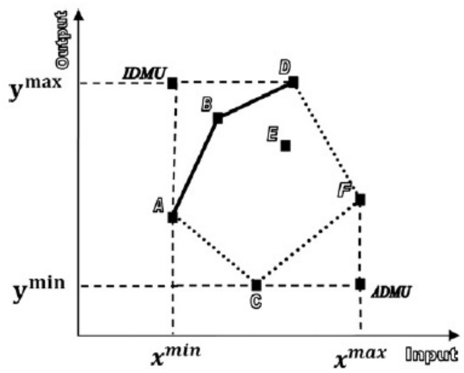

Fig.1 CHD model

In CHD model, ρ^*_phas not scalar measure. With increase slack variables ρ^*_P is decrease. CHD model has not oriented.Fig. 1 shows CHD model has not feasibility axiom and constant returns-to-scale axiom. That is, in CHD model any observation with input no less than x in any component and with output no greater than y in any component is not feasible.CHD model have three axiom principles: inclusion principle, convexity principle and smallest set that satisfying the principles inclusion and convexity.

3 CHD model with interval data

We call interval data as follows:

$$[a, b] = \{x \mid a \le x \le b\} \qquad (7)$$

When it isn't possible to determine exact data, rather the only inputs and outputs changing rang is determined, it must be used of models with interval data. CHD model is presented to calculate the exact data. In this paper, we are studied CHD model with interval data by first method and second method. The obtained efficiency of these two methods is interval. Two methods are presented to calculate the interval data efficiency. Lower bound and upper bound is obtaining for DMU under evaluation in each method.

4 First method for efficiency interval in CHD model

A.efficiency lower bound of DMUs:

We consider the most pessimistic for DMU_P under evaluation. In other words DMU_Phave the most input and the least output. We consider the most optimistic for other DMUs. That is DMU_j , (j ≠ p) having the least input and the most output. Efficiency lower bound evaluation of DMUs is following form in the CHD model:

$$\begin{aligned}
& \min \quad H_p^L = 1 - \frac{\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-}{\alpha + \varepsilon} \\
& subjuct\ \ to \qquad\qquad (8) \\
& \sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j y_{rj}^U + \lambda_p y_{rp}^L - s_r^+ = y_{rp}^L, \quad r = 1, \ldots, s \\
& \sum_{\substack{j=1 \\ j \neq p}}^{n} \lambda_j x_{ij}^L + \lambda_p x_{ip}^U + s_r^- = x_{ip}^U, \quad i = 1, \ldots, m \\
& \sum_{j=1}^{n} \lambda_j = 1 \\
& \lambda_j \ge \circ, \quad j = 1, \ldots, n, \quad s_r^+, \ s_i^- \ge \circ
\end{aligned}$$

In this model, y_{rp}^L is the least output of DMU_P and x_{ip}^U is the most input of DMU_p . y_{rj}^U is the most output of DMU_j , (j ≠ p) and x_{ij}^L is the least input DMU_j , (j ≠ p).

B. efficiency upper bound of DMUs:
We consider the most optimistic for DMU_P under evaluation. In other words DMU_P have the least input and the most output. We consider the most pessimistic for other DMUs. That is DMU_j , ($j \neq p$) having the most input and the least output. Efficiency upper bound evaluation of DMUs is following form in the CHD model:

$$\text{Min} \quad H_p^L = 1-\frac{\sum_{r=1}^{s} s_r^+ + \sum_{i=1}^{m} s_i^-}{\alpha+\varepsilon}$$

$$s.t. \tag{9}$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j y_{rj}^L + \lambda_p y_{rp}^U - s_r^+ = y_{rp}^U, \ \text{r=1,...,s}$$

$$\sum_{\substack{j=1 \\ j\neq p}}^{n} \lambda_j x_{ij}^U + \lambda_p x_{ip}^L + s_r^- = x_{ip}^L, \ \text{i=1,...,m}$$

$$\sum_{j=1}^{n} \lambda_j = 1$$

$$\lambda_j \geq \circ, \ \text{j=1,...,n},$$

$$s_r^+, \ s_i^- \geq \circ$$

In this model, y_{rp}^U isthe most output of DMU_P and x_{ip}^L is the least input ofDMU_P. y_{rj}^L is the least output of DMU_j , ($j \neq p$) and x_{ij}^U is the most input DMU_j , ($j \neq p$).
The obtained efficiency upper bound and the obtained efficiency lower bound are building interval efficiency of DMU.

5 Second method for efficiency interval in CHD model

In this method, the all DMUs are considered in similar state. Therefore,DMU_P isn't separable of other DMUs. But Efficiency upper bound evaluation of DMUs and Efficiency lower bound evaluation of DMUs is isolated. In order to use of this method, we are employ the dual of CHD model. The dual of CHD model is as follows:

$$
\begin{aligned}
& Min \quad \sum_{i=1}^{m} v_i x_{ip} - \sum_{r=1}^{s} u_r y_{rp} + u_o \\
& s.t. \\
& \sum_{i=1}^{m} v_i x_{ij} - \sum_{r=1}^{s} u_r y_{rj} + u_o \geq \circ \\
& v_i \geq \frac{1}{\alpha + \varepsilon} \\
& u_r \geq \frac{1}{\alpha + \varepsilon} \\
& u_o \ \text{is free}
\end{aligned} \tag{10}
$$

A. efficiency lower bound of DMUs:
Lower bound obtains in the pessimistic state which all DMUs have the most input and the least output. Efficiency lower bound evaluation of DMUs is following form in the CHD model:

$$
\begin{aligned}
& Min \quad \sum_{i=1}^{m} v_i x_{ip}^{U} - \sum_{r=1}^{s} u_r y_{rp}^{L} + u_o \\
& s.t. \\
& \sum_{i=1}^{m} v_i x_{ij}^{U} - \sum_{r=1}^{s} u_r y_{rj}^{L} + u_o \geq \circ \\
& v_i \geq \frac{1}{\alpha + \varepsilon} \\
& u_r \geq \frac{1}{\alpha + \varepsilon} \\
& u_o \ \text{is free}
\end{aligned} \tag{11}
$$

B. efficiency upper bound of DMUs:
Upper bound obtains in the optimistic state which all DMUs have the least input and the most output. Efficiency lower bound evaluation of DMUs is following form in the CHD model:

$$
\begin{aligned}
& Min \quad \sum_{i=1}^{m} v_i x_{ip}^{L} - \sum_{r=1}^{s} u_r y_{rp}^{U} + u_o \\
& s.t. \\
& \sum_{i=1}^{m} v_i x_{ij}^{L} - \sum_{r=1}^{s} u_r y_{rj}^{U} + u_o \geq \circ \\
& v_i \geq \frac{1}{\alpha + \varepsilon} \\
& u_r \geq \frac{1}{\alpha + \varepsilon} \\
& u_o \ \text{is free}
\end{aligned} \tag{12}
$$

6 Numerical example

Consider six DMUs which consume two inputs (x_1, x_2) to produce two outputs (y_1, y_2). Inputs and outputs of DMUs is interval. Table 1 shows inputs and outputs any DMU.

Table 1 Inputs and outputs DMUs

DMU				
A	[12,15]	[0.21,0.48]	[138,144]	[21,22]
B	[10,17]	[0,1.7]	[143,159]	[28,35]
C	[4,12]	[0.16,0.35]	[157,198]	[21,29]
D	[19,22]	[0.12,0.19]	[158,181]	[21,25]
E	[14,15]	[0.06,0.09]	[157,161]	[28,40]
F	[8,10]	[0.08,0.1]	[150,180]	[36,39]

Efficiency evaluation DMUs is calculated in CHD model with employ both first method and second method. Using the interval models, we obtain the rating results listed in the table 2. Table 2 shows the results of efficiency in CHD model with first method and second method.

Table 2 Optimal values CHD model

DMU	Efficiency in first method	Efficiency in second method
A	[0.798,0.938]	[0.156,0.798]
B	[0.147,0.385]	[0.385,1.000]
C	[1.000,1.000]	[1.000,1.000]
D	[0.218,1.000]	[0.183,0.343]
E	[1.000,1.000]	[1.000,1.000]
F	[1.000,1.000]	[1.000,1.000]

The obtained results show that efficiency of DMUs is imprecise data and bounded. The efficient DMUs in both methods are same. As can be seen from Table 2 that due to the use of variable production frontiers to measure the efficiencies of different DMUs.

Using by models (8) and (11), we obtain the lower bound efficiencies of each DMU. Using by models (9) and (12), we obtain the upper bound efficiencies of each DMU. The efficient DMUs in both methods are DMUC, DMUE and DMUF. Since CHD model isbased on the slacks therefore increase slack variables ρ^*_P is decrease. Then DMU A, DMU B and DMU D for reach to efficient frontier are need its slack variables decrease while reach to zero.

7 Conclusions

In this paper, we supposed PPS is as $T_{CHD} = \left\{(x, y) \middle| x = X\lambda,\ y = Y\lambda, 1\lambda = 1,\ \lambda \geq \circ\right\}$ and also analyzed the CHD model. Then, we applied two methods, the first and second method, efficiency evaluation of DMUs explained for interval data in the CHD model. Efficiency evaluation of DMUs in CHD model is performed on the convex hull. We used the first and second method applied in CHD model for interval data. The obtained efficiency is interval in both methods. The obtained efficiency upper bound and the obtained efficiency lower bound are building the interval efficiency of DMU. Efficiency frontier is changing in the first method and efficiency for any DMU under evaluation is obtaining on separable frontier. But in the second method the only one frontier is performing for all DMUs under evaluation.

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